Combinatorics on q-deformed rationals

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This talk does not concern Commutative Algebra (at least, directly), but has a similar flavor to *Combinatorial* Commutative Algebra. I would like to ask for your permission to present it here. The talk is based on the joint works with Takeyoshi Kogiso, Kengo Miyamoto. Xin Ren (任鑫) and Michihisa Wakui.

For $n \in \mathbb{N}$, the q-deformation (q-analog) $[n]_q = 1 + q + \cdots + q^{n-1}$ is a classical subject of mathematics. Recently, Morier-Genoud and Ovsienko [4] introduced a q-deformation of a rational number (irreducible fraction) $\frac{r}{s}$

$$\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}_{\frac{r}{s}}(q)}{\mathcal{S}_{\frac{r}{s}}(q)} \qquad (\mathcal{R}_{\frac{r}{s}}(q) \in \mathbb{Z}[q^{\pm 1}], \mathcal{S}_{\frac{r}{s}}(q) \in \mathbb{Z}[q] \text{ with } \mathcal{R}_{\frac{r}{s}}(1) = r, \mathcal{S}_{\frac{r}{s}}(1) = s\ ).$$

This notion is related to many directions including 2-Calabi-Yau categories, Jones polynomials of rational knots, and combinatorics of posets.

Let me show a few examples. For all $n \in \mathbb{Z}$ we have $\left[\frac{n}{1}\right]_q = \frac{[n]_q}{1}$. We also have $\left[\frac{7}{5}\right]_q = \dfrac{q^4 + 2q^3 + 2q^2 + q + 1}{q^3 + 2q^2 + q + 1}$ and $\left[\frac{8}{5}\right]_q = \dfrac{q^4 + 2q^3 + 2q^2 + 2q + 1}{q^3 + q^2 + 2q + 1}$.

There are several ways to construct/compute $[\frac{r}{s}]_q$. The most sophisticated way uses the *the q-deformed modular group* $\mathrm{PSL}_q(2, \mathbb{Z})$, but we introduce a naive one here, while it only works for $\frac{r}{s} > 1$. For an irreducible fraction $\frac{r}{s} > 1$, we have

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots\, a_{k-1} + \cfrac{1}{a_k}}}$$

for some $a_1, \ldots, a_k \in \mathbb{Z}_{>0}$. We write $[a_1, \ldots, a_k]$ for this expansion. Since $[a_1, \ldots, a_k, 1] = [a_1, \ldots, a_k + 1]$, we can always take that the length k is even. For $\frac{r}{s} = [a_1, \ldots, a_{2m}] > 1$, we have the posets $\mathcal{N}_{\frac{r}{s}}$ and $\mathcal{D}_{\frac{r}{s}}$ whose Hasse diagram are the following.

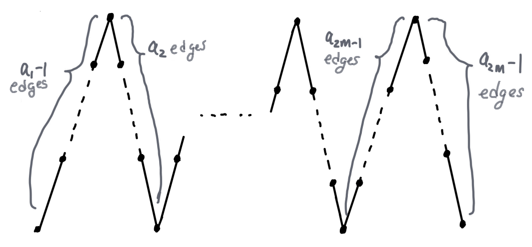

FIGURE 1. The poset $\mathcal{N}_{\frac{r}{s}}$

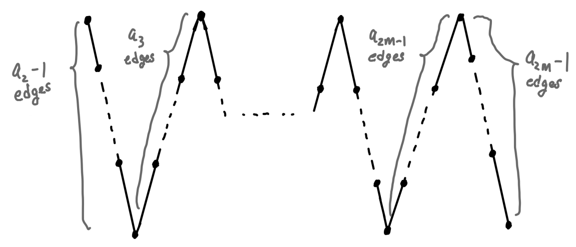

FIGURE 2. The poset $\mathcal{D}_{\frac{r}{s}}$

For a finite poset P, set $\rho_P(i) := \#\{I \subset P \mid I \text{ is a lower ideal with } \#I = i\}$ and $\mathsf{rk}(P, q) := \sum_{i \geq 0} \rho_P(i) \cdot q^i$. In this situation, Morier-Genoud and Ovsienko [4] showed that $\mathcal{R}_{\frac{r}{s}}(q) = \mathsf{rk}(\mathcal{N}_{\frac{r}{s}}, q)$ and $\mathcal{S}_{\frac{r}{s}}(q) = \mathsf{rk}(\mathcal{D}_{\frac{r}{s}}, q)$.

For $f(q) = a_0 + a_1 q + \cdots + a_n q^n \in \mathbb{Z}[q]$ with $a_n \neq 0$, set $f(q)^\vee := a_n + a_{n-1}q + \cdots + a_1 q^{n-1} + a_0 q^n$. We say $f(q) \in \mathbb{Z}[q]$ is *palindromic*, if $f(q) = f(q)^\vee$.

Lemma 1 (Leclere, Morier-Genoud, Ovsienko)**.** For $\alpha \in \mathbb{Q}$ and $n \in \mathbb{Z}$, $[\alpha + n]_q = q^n[\alpha]_q + [n]_q$ holds. Hence, $r \equiv r' \pmod{s}$ implies $\mathcal{S}_{\frac{r}{s}}(q) = \mathcal{S}_{\frac{r'}{s}}(q)$.

In the following, we state our results for $\mathcal{S}_{\frac{r}{s}}(q)$, but similar hold for $\mathcal{R}_{\frac{r}{s}}(q)$.

Theorem 2 (Kogiso, Miyamoto, Ren, Wakui & Y.)**.** The following hold.

(1) If $r + r' \equiv 0 \pmod{s}$, then $\mathcal{S}_{\frac{r}{s}}(q) = \mathcal{S}_{\frac{r'}{s}}(q)^\vee$. Similarly, if $rr' \equiv 1 \pmod{s}$, then $\mathcal{S}_{\frac{r}{s}}(q) = \mathcal{S}_{\frac{r'}{s}}(q)^\vee$. Hence, if $rr' \equiv -1 \pmod{s}$, then $\mathcal{S}_{\frac{r}{s}}(q) = \mathcal{S}_{\frac{r'}{s}}(q)$.

(2) $\mathcal{S}_{\frac{r}{s}}(q)$ is palindromic $\iff r^2 \equiv 1 \pmod{s}$.

Morier-Genoud and Ovsienko [4] showed that $\mathcal{S}_{\frac{r}{s}}(-1) \in \{0, \pm 1\}$, and s is even if and only if $\mathcal{S}_{\frac{r}{s}}(-1) = 0$. Then next extends this result.

Theorem 3 (KMRWY)**.** The following hold.

(1) Set $\omega = \frac{-1+\sqrt{-3}}{2}$. We have $\mathcal{S}_{\frac{r}{s}}(\omega) \in \{0, \pm 1, \pm\omega, \pm\omega^2\}$. In particular, s is a multiple of 3 $\iff \mathcal{S}_{\frac{r}{s}}(\omega) = 0$.

(2) For $i = \sqrt{-1}$, we have $\mathcal{S}_{\frac{r}{s}}(i) \in \{0, \pm 1, \pm i, \pm(1+i), \pm(1-i)\}$. In particular, s is a multiple of 4 $\iff \mathcal{S}_{\frac{r}{s}}(i) = 0$.

The above theorem cannot be extended further. For $n \geq 5$, let ζ_n be the primitive n-th root of unity. Even if s is a multiple of n, $\mathcal{S}_{\frac{r}{s}}(\zeta_n) \neq 0$ happens quite often.

Conjecture 4 (KMRWY)**.** If s is prime, $\mathcal{S}_{\frac{r}{s}}(q)$ is irreducible over $\mathbb{Q}$.

J.H. Conway associated a rational number $\frac{r}{s} > 1$ with the "rational" link $L(\alpha)$ (not "1 to 1"). The *normalized Jones polynomial* $J_{\frac{r}{s}}(q)$ of $L(\frac{r}{s})$ turns out to be a "variant" of the polynomials $\mathcal{R}_{\frac{r}{s}}(q), \mathcal{S}_{\frac{r}{s}}(q)$ with $J_{\frac{r}{s}}(1) = r$, $J_{\frac{r}{s}}(0) = 1$. See [3, 4].

Theorem 5 (Ren & Y.)**.** The following hold.

(1) $J_{\frac{r}{s}}(q)$ is palindromic $\iff r^2 \equiv -1 \pmod{s}$.

(2) If $J_{\frac{r}{s}}(q)$ is not palindromic, the polynomial $(J_{\frac{r}{s}}(q) - J_{\frac{r}{s}}(q)^\vee)/(1-q)$ is palindromic, monic, and sign coherent.

Finally, we discuss the irreducible decomposition of $J_{\frac{r}{s}}(q)$ in $\mathbb{Z}[q]$ (e.g., a variant of Conjecture 4).

References

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